Math 251
Exam IV v1
Spring 2016

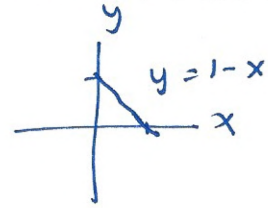
Show All Work

name Key

1.) **Set up** $\iiint_E x^2 dV$ where E is the solid tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1)

$x+y+z=1$

base



$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 \, dz \, dy \, dx$$

2.) Find $\iiint_E x^2 dV$ where E is the region in the first octant below the plane $z + y = 3$ and inside the cylinder $x^2 + y^2 = 1$.

$$\int_0^{\pi/2} \int_0^1 \int_0^{3-y} x^2 \, dz \cdot r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 \int_0^{3-r\sin\theta} r^2\cos^2\theta \, dz \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^2\cos^2\theta \, z \Big|_{z=0}^{z=3-r\sin\theta} \cdot r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 r^2\cos^2\theta \cdot (3 - r\sin\theta) \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 (3r^3\cos^2\theta - r^4\cos^2\theta\sin\theta) \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{3r^4}{4}\cos^2\theta - \frac{r^5}{5}\cos^2\theta\sin\theta) \Big|_0^1 d\theta = \int_0^{\pi/2} \left(\frac{3}{4}\cos^2\theta - \frac{1}{5}\cos^2\theta\sin\theta\right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{3}{8}(1+\cos 2\theta) - \frac{1}{5}\cos^2\theta\sin\theta\right) d\theta$$

$$= \frac{3}{8}\left(\theta + \frac{\sin 2\theta}{2}\right) - \frac{1}{5}\left(\frac{-\cos^3\theta}{3}\right) \Big|_0^{\pi/2}$$

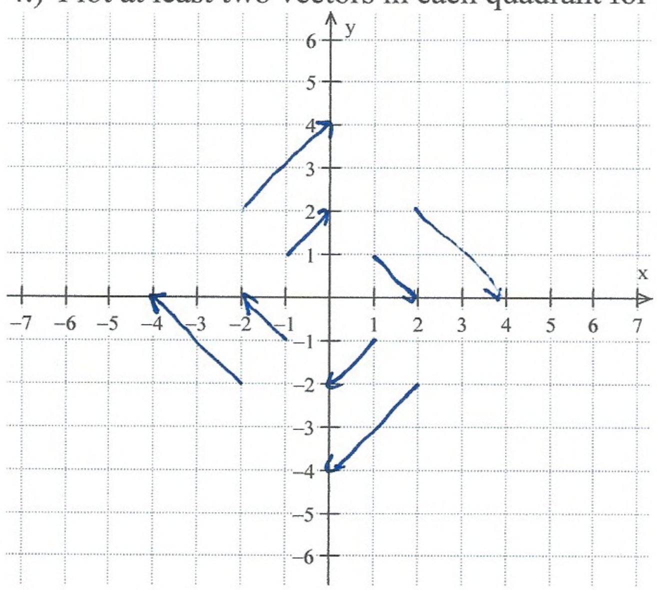
$$= \left[\frac{3}{8}\left(\frac{\pi}{2} + 0\right) + \frac{1}{15}\cdot 0\right] - \left[\frac{3}{8}(0+0) + \frac{1}{15}\cdot 1\right]$$

$$= \frac{3\pi}{16} - \frac{1}{15}$$

3.) **Set up** $\iiint_H z\sqrt{x^2+y^2+z^2}\,dV$ where H is the solid hemisphere that lies **below** the xy-plane and has center origin and radius 3.

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{3} (\rho\cos\varphi)\cdot\rho\cdot\rho^2\sin\varphi \; d\rho\, d\varphi\, d\theta$$

4.) Plot at least two vectors in each quadrant for the vector field **F** = < y, - x >



point	F
(1,1)	<1,-1>
(2,2)	<2,-2>
(1,-1)	<-1,-1>
(2,-2)	<-2,-2>
(-1,-1)	<-1,1>
(-2,-2)	<-2,2>
(-1,1)	<1,1>
(-2,2)	<2,2>

5.) Evaluate the following line integrals.

a) $\int_C -y\,dx + xy\,dy$ where C is the line segment from (1, 2) to (-1, 6).

$r(t) = \langle -2t+1,\ 4t+2\rangle$

$r'(t) = \langle -2, 4\rangle$

$$\int_0^1 \left[-(4t+2)\cdot(-2) + (-2t+1)(4t+2)\cdot 4\right] dt$$

$$= \int_0^1 \left[8t+4 + 4(-8t^2+2)\right] dt = \int_0^1 -32t^2+8t+12\, dt$$

$$= -\frac{32}{3}t^3 + 4t^2 + 12t \Big|_0^1 = -\frac{32}{3} + 4 + 12 = -\frac{32}{3} + \frac{12}{3} + \frac{36}{3} = \frac{16}{3}$$

b) $\int_C x^2 ds$ where C is the part of the circle $x^2 + y^2 = 4$ that lies in the first quadrant.

$x = 2\cos t$ $\quad y = 2\sin t$ $\quad 0 \le t \le \pi/2$

$\frac{dx}{dt} = -2\sin t$ $\quad \frac{dy}{dt} = 2\cos t$

$$\int_0^{\pi/2} 4\cos^2 t \cdot \sqrt{4\sin^2 t + 4\cos^2 t}\, dt = \int_0^{\pi/2} 4\cos^2 t \cdot 2\, dt$$

$$= 8\int_0^{\pi/2} \cos^2 t\, dt = 8\int_0^{\pi/2} \frac{1+\cos 2t}{2}\, dt = 4\int_0^{\pi/2} (1+\cos 2t)\, dt$$

$$= 4\left[t + \frac{\sin 2t}{2}\right]\Big|_0^{\pi/2} = \left[4\left(\frac{\pi}{2}+0\right) - 4(0+0)\right] = 2\pi$$

6.) A)Determine whether the function $\mathbf{F}(x, y) = \langle ye^x + 3x^2, e^x + \sin y \rangle$ is conservative. If it is, find its potential function.

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = e^x$$ conservative ✓

$$f(x,y) = ye^x + x^3 - \cos y + K$$

7.) Let F(x, y) = <P(x, y), Q(x, y)>

a) Under what conditions is $\int_C \boldsymbol{F}\cdot \boldsymbol{dr} = \int_C P(x,y)dx + Q(x,y)dy$ independent of path?

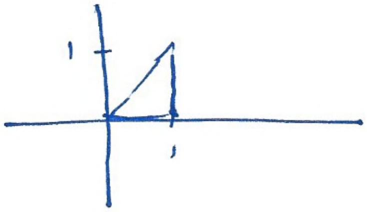
If its conservative $\left(\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}\right)$

8.) Find the work done by **F**(x, y) = $\langle x^2 + y^2, 2xy\rangle$ where C is the top half of the circle centered at the origin with radius 2, from (2, 0) to (-2, 0).

note $\frac{\partial Q}{\partial x} = 2y = \frac{\partial P}{\partial y}$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{x^3}{3} + xy^2 \Big|_{(2,0)}^{(-2,0)} = \left(\frac{-8}{3}+0\right) - \left(\frac{8}{3}+0\right) = \frac{-16}{3}$$

9.) Find $\int_C e^y\, dx + x^2 dy$ where C is the triangle from (0,0) to (1,0) to (1, 1) to (0, 0)



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GREENS THEOREM APPLIES

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^x (2x - e^y)\, dy\, dx = \int_0^1 2xy - e^y \Big|_0^x dx$$

$$= \int_0^1 (2x^2 - e^x) - (0 - e^0)\, dx = \int_0^1 (2x^2 - e^x + 1)\, dx$$

$$= \frac{2x^3}{3} - e^x + x \Big|_0^1 = \left(\frac{2}{3} - e + 1\right) - (0 - 1 + 0)$$

$$= \frac{2}{3} - e + 1 + 1 = \frac{2}{3} + 2 - e = \frac{8}{3} - e$$